Actions on the Hilbert cube

To Cor Baayen, at the occasion of his retirement.

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We provide a negative answer to Problem 933 in the "Open Problems in Topology Book".

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1 Introduction

Let Q denote the Hilbert cube $\prod_{i=1}^{\infty}[-1,1]_i$. In the "Open Problems in Topology Book", West [2] asks the following (Problem #933):

> *Let the compact Lie group G act semifreely on Q in two ways such that their fixed point sets are identical. If the orbit spaces are* **ANR***'s, are the actions conjugate?*

The aim of this note is to present a counterexample to this problem. For all undefined notions we refer to [1].

2 The Example

Let G be a group and let $\pi: G \times X \to X$ be an action from G on X. Define $\mathcal{F}ix(G) = \{x \in X : (\forall g \in G)(\pi(g,x) = x)\}$. It is clear that $\mathcal{F}ix(G)$ is a closed subset of X: it is called the *fixed-point set* of G The action π is called *semifree* if it is free off $\mathcal{F}ix(G)$, i.e., if $x \in X \setminus \mathcal{F}ix(G)$ and $\pi(g,x) = x$ for some $g \in G$ then g is the identity element of G. The space of orbits of the action π will be denoted by X/G. Let $\mathbb{I}$ denote the interval $[0,1]$.

Let G denote the compact Lie group $\mathbb{T} \times \mathbb{Z}_2$, where $\mathbb{T}$ denotes the circle group. We identify $\mathbb{Z}_2$ and the subgroup $\{-1,1\}$ of $\mathbb{T}$. In addition, D denotes $\{z \in \mathbb{C} : |z| \leq 1\}$. We let G act on $D \times D$ in the obvious way:

$$\big((g,\varepsilon),(x,y)\big) \mapsto (g \cdot x, \varepsilon \cdot y) \qquad (g \in \mathbb{T}, \varepsilon \in \{-1,1\}, x, y \in D),$$

where "$\cdot$" means complex multiplication. Observe that this action is semifree, and that its fixed-point set contains the point $(0,0)$ only. Also, observe that $(D \times D)/G \approx \mathbb{I} \times D$.

LEMMA 2.1 *Let H denote either G or $\mathbb{T}$. There is a semifree action of H on $Q \times \mathbb{I}$ having $Q \times \{0\}$ as its fixed-point set. Moreover, $(Q \times \mathbb{I})/G$ and Q are homeomorphic.*

PROOF. We will only prove the lemma for G since the proof for $\mathbb{T}$ is entirely similar. We first let G act on $X = D \times D \times Q$ as follows:

$$\big((g,\varepsilon),(x,y,z)\big) \mapsto (g \cdot x, \varepsilon \cdot y, z) \qquad (g \in \mathbb{T}, \varepsilon \in \{-1,1\}, x, y \in D, z \in Q).$$

This action is semifree and its fixed-point set is equal to $\{(0,0)\} \times Q$. Also observe that $X/G \approx \mathbb{I} \times D \times Q$.

We now let G act coordinatewise on the infinite product X^∞. This action is again semifree, having the diagonal $\triangle$ of $\{(0,0)\} \times Q$ in X^∞ as its fixed-point set. Also, X^∞/G is homeomorphic to $(I \times D \times Q)^\infty \approx Q$. Since $\triangle$ projects onto a proper subset of X in every coordinate direction of X^∞, it is a Z-set. Since $X^\infty \approx Q$ there consequently is a homeomorphism of pairs $(X^\infty, \triangle) \to (Q \times \mathbb{I}, Q \times \{0\})$. We are done.

We will now describe two actions of G on $Q \times [-1,1]$. By Lemma 2.1 there is a semifree action $\alpha_r: \mathbb{T} \times Q \times \mathbb{I} \to Q \times \mathbb{I}$ having $Q \times \{0\}$ as its fixed point set, while moreover $Q \times \mathbb{I}/G \approx Q$. We let $\mathbb{T}$ act on $Q \times [-1,0]$ as follows:

$$\big(z,(q,t)\big) \mapsto (\bar{q}, s) \quad \text{iff} \quad \alpha_r\big(z,(q,-t)\big) = (\bar{q}, -s).$$

We will denote this action by α_l. So $\alpha = \alpha_l \cup \alpha_r$ is an action of $\mathbb{T}$ onto $Q \times [-1,1]$, having $Q \times \{0\}$ as its fixed-point set. Now define $\bar{\alpha}: G \times (Q \times [-1,1]) \to Q \times [-1,1]$ as follows:

$$\bar{\alpha}\big((z,\varepsilon),(q,t)\big) = \begin{cases} \alpha\big(z,(q,t)\big), & (\varepsilon = 1), \\ \alpha\big(z,(q,-t)\big), & (\varepsilon = -1). \end{cases}$$

Then $\bar{\alpha}$ is a semifree action of G onto $Q \times [-1,1]$ having $Q \times \{0\}$ as its fixed-point set, while moreover $(Q \times [-1,1])/\bar{\alpha} \approx Q$. Observe the following triviality.

LEMMA 2.2 *If $A \subseteq Q \times [-1,1]$ is $\bar{\alpha}$-invariant such that A is not contained in $Q \times \{0\}$, then A intersects $Q \times (0,1]$ as well as $Q \times [-1,0)$.*

We will now describe the second action on $Q \times [-1, 1]$. By Lemma 2.1 there is a semifree action $\beta_r: G \times Q \times \mathbb{I} \to Q \times \mathbb{I}$ having $Q \times \{0\}$ as its fixed point set, while moreover $Q \times \mathbb{I}/G \approx Q$. Construct β_l from β_r in the same way we constructed α_l from α_r. Then $\beta = \beta_l \cup \beta_r$ is a semifree action from G onto $Q \times [-1, 1]$ having $Q \times \{0\}$ as its fixed-point set. Moreover, $(Q \times \mathbb{I})/\beta$ is the union of two Hilbert cubes, meeting in a third Hilbert cube, hence is an **AR**. (It can be shown that $(Q \times \mathbb{I})/\beta \approx Q$.)

Now assume that the two axions $\bar{\alpha}$ and β are conjugate. Let $\tau: Q \times [-1, 1] \to Q \times [-1, 1]$ be a homeomorphism such that for every $g \in G$, $\beta(g) = \tau^{-1} \circ \bar{\alpha}(g) \circ \tau$. Then $\tau(Q \times (0, 1])$ is a connected $\bar{\alpha}$-invariant subset of $Q \times [-1, 1]$ which misses $Q \times \{0\}$. This contradicts Lemma 2.2.

References

1. J. van Mill. *Infinite-Dimensional Topology: prerequisites and introduction.* North-Holland Publishing Company, Amsterdam, 1989.
2. J. E. West. Open Problems in Infinite Dimensional Topology. In J. van Mill and G. M. Reed, editors, *Open Problems in Topology*, pages 523–597, North-Holland Publishing Company, Amsterdam, 1990.